Capital Markets and Portfolio Theory

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1 Expected utility theory

1.1 The plan

We start by considering an individuals preferences in a world without uncertainty. We need to be able to make choices between consumption bundles. Examples of consumption bundles are: 2 apples, 2 oranges, a \$1.00 today plus \$2.00 in one year. From here on in will assume that all consumption items are valued in money (dollars). If there is no inflation then this assumption is relatively harmless. This reduces consumption bundles to money consumed at various times. In other worlds a consumption bundle has been simplified to a vector of dollar amounts.

Example 1.1 Suppose we consider consumption bundles in $\mathbb{R}^2_+$ (2-dimensional positive real numbers) and let time 0 be now and time 1 a year hence. If $c_1 = (\$100, \$150)$ and $c_2 = (\$125, \$125)$ then preferring c_1 to c_2 would be a preference for \$100 today and \$150 in a year to \$125 in each year.

If we are going to model choices in an uncertain world, then we need a way to model uncertainty. The the state of nature is denoted by ω. The collection of states of nature is called Ω. The state of nature ω tells us everything we need to know about the world. For example ω tells us, interest rates in the United States, unemployment levels in Europe, world population, which countries are at war, and everything else you can think of. To build useful models ω really needs to be a function of time, discrete or continuous. Although this extension to mutiperiods is straight forward it is cumbersome, hence we will restrict our discussion to the state of the world at a single point in time. Obviously, Ω is a large set, in many cases it is infinite (e.g. if consumption goods are infinitely divisible).

We will model a consumption schedule as a random variable on Ω. It is easiest to illustrate with a simple example. Suppose that there are three possible states of nature $\Omega = \{\omega_1, \omega_2, \omega_3\}$ then a consumption schedule or random variable $x(\omega)$ might be

	ω_1	ω_2	ω_3
$x(\omega)$	-\$5	\$0	\$20

Suppose we had a function $U(x(\omega))$ that represented preferences, in other words $x(\omega)$ is preferred to $y(\omega)$ if and only iff $U(x) > U(y)$ (I'll write x for $x(\omega)$ to simplify notation, but remember that x is a function of ω). This would allow us to compare random variables by comparing numbers, this is a good thing. $U(x(\omega))$ is a function of a function, a complex object. Since we have taken x to be dollars, it is not all that difficult to specify $U(x)$. However, to specify $U(x(\omega))$ is almost impossible. U would be much easier to work with if (and specify) if it could be represented as an expectation or average of some other function u where u is a function of a positive real number . We are looking for a function u such that

$$U(x) = E[u(x)]$$

or if Ω is finite

$$U(x) = \sum_{\omega \in \Omega} p(\omega)\, u(x(\omega))$$

we say that U has an expected utility representation. If all of this can be done then all that we need to know to model consumption and investment choices is the individuals function u and the probability distribution of any consumption schedule (i.e. asset, labor, etc.). We now proceed to show what assumptions need to be made in order for the above program to work. In other words we examine when it is reasonable for an expected utility representation to exist.

1.2 Preferences

We start off very generally by defining a preference relation on some abstract set of alternatives, say X. In practice we will always take X to be either vectors of consumption bundles indexed by time or random consumption schedules. For elements $x, y \in X$ we write $x \succeq y$ and say x is preferred to y or x is at least as good as y. We don't claim to prove anything in its greatest generality or with the smallest number of assumptions, for this see Kreps (1988) or Fishburn (1970).

Definition 1.1 *A binary relation $\succeq$ is called a preference relation if it is*

1. *Complete: for all $x, y \in X$, we have $x \succeq y$ or $y \succeq x$ or both, and*

2. *Transitive: for all $x, y, z \in X$, if $x \succeq y$ and $y \succeq z$ then $x \succeq z$.*

We also define *strict preference* (written $x \succ y$) as $x \succeq y$ and $y \not\succeq x$. and *indifference* (written $x \sim y$) as $x \succeq y$ and $y \succeq x$.

Exercise 1.1 Prove that $\succ$ is irreflexive, i.e. for all $x \in X$ it is not that case that $x \succ x$.

Exercise 1.2 Many authors also include reflexivity ($x \succeq x$ for all x) as part of the definition of preference relation, but this is redundant. Prove it.

Remark 1.1 It is easy to show, that a strict preference relation is transitive and that indifference is an equivalence relation, i.e. $\sim$ is reflexive ($x \sim x$), symmetric ($x \sim y$ implies $y \sim x$), and transitive.

Remark 1.2 We could just as easily have started with a strict preference relation that is asymmetric ($x \succ y$ implies $y \not\succ x$) and negatively transitive ($x \not\succ y$ and $y \not\succ z$ implies $x \not\succ z$). We could then derive our "weak" preference $\succeq$ relation with the properties above. You may want to try this if you are interested.

1.3 Ordinal Utility

We would like to express $\succeq$ numerically, i.e. a function $U : X \to \mathbb{R}$ (the real line) such that

$$U(x) \geq U(y) \text{ implies } x \succeq y \tag{1.1}$$

We call U an ordinal utility function representing $\succeq$. Clearly for a given preference relation $\succeq$, there is no unique utility function representing $\succeq$. For any monotonic function on the real line f, $f \circ U$ (the composition of f with U) is also a utility function representing $\succeq$. It is only the ranking of the alternatives that matters, it is for this reason that U is referred to as an ordinal utility function.

Proposition 1.1 *Suppose that we are given a utility function U, if we define $\succeq$ by (1.1) then $\succeq$ is a preference relation.*

Exercise 1.3 Prove proposition 1.1.

It is the converse to this proposition that we are interested in, unfortunately it is not always true.

Proposition 1.2 *If X is countable then any preference relation can be represented by some utility function U. If X is a subset of a separable metric space and $\succeq$ is continuous (defined appropriately) then $\succeq$ an be represented by some utility function U.*

Proof: See Kreps (1988). □

Example 1.2 (Lexicographic preference relation) This example presents a preference relation that has no ordinal utility representation. Take X to be the set of order pairs of real numbers, $x = (x_1, x_2) \in X$. We define the lexicographic preference relation as $x \succeq y$ if either $\{x_1 > y_1\}$ or $\{x_1 = y_1 \text{ and } x_2 \geq y_2\}$. It is called lexicographic since this is the way a dictionary is ordered, i.e. alphabetically. It can be shown that there is no utility function that represents this preference relation. We will not prove this, but it can be seen intuitively as follows. No two distinct elements of X are indifferent, therefore we have two dimensions of distinct indifference sets. To construct a utility function each indifference set must be assigned a number from the one dimensional real line, that preserves order.

Exercise 1.4 Show that the lexicographic preference relation is complete and transitive.

1.4 Expected Utility

The previous section demonstrates that the notion of *preference relation* can always be quantified in a certain sense. If we can say for any pair of alternatives which we prefer, then we can also assign a utility to each alternative in a way that treats greater utility as preferable. We now change our perspective slightly by considering the space of consumption schedules. In other words we take X to be the space of consumption schedules or random variables on Ω. Since any random variable induces a unique distribution on the real line we can equivalently take as X the space P of probability distributions on the line. We will use the terms probability distribution, gamble, and lottery interchangeably to denote the elements of P. Our aim is to study preference relations on P. At first blush this seems like an odd thing to want to do, but we will see it is quite natural. This is useful, for example, in the investment context. Consider an investor with $100 and two investment alternatives: a stock ABC and a bank account bearing no interest. Suppose the stock is $1 per share now an tomorrow will be worth either $0 or $2. Then if the investor buys x shares of ABC today, tomorrow he will have $10\bar{0} + x$ or $100 - x$. The investor has 100 distinct investment decisions and therefore has 200 possible outcomes

50%	100	101	102	...	200	up
50%	100	99	98	...	0	down

The investor cannot choose what outcome he wants (otherwise he would presumably choose $200). Instead he can choose 100 different lotteries:

$$\begin{aligned}
\ell_1 &= 100\% : \$100 \\
\ell_2 &= 50\% : \$101, 50\% : \$99 \\
\ell_3 &= 50\% : \$102, 50\% : 98 \\
&\vdots \\
\ell_{100} &= \dots
\end{aligned}$$

Note that not all lotteries are available. This is not a problem and is a feature of the investment. The point is there are a number of investment decisions available to an investor, each yielding a different lottery on a fixed set of financial outcomes ($0-$200). Which lottery does the investor prefer.

By the results of the previous section we know that there exists a non-unique utility function U that represents any continuous preference relation on P. We want to know if we can find another utility function u on the space of consumption bundles so that for $p, q \in P$, $p \succeq q$ if and only if the expected value of u under p is greater than or equal to its expectation under q, $U(p) = E^p[u(x)] \geq E^q[u(x)] = U(q)$ or,

$$\int_{-\infty}^{\infty} u(x)dp(x) \geq \int_{-\infty}^{\infty} u(x)dq(x)$$

If such a u exists we say $\succeq$ has an expected utility representation and then we call it a von Neumann-Morgenstern utility function after its originators, henceforth when we say utility function we mean von Neumann-Morgenstern utility function, otherwise we will say ordinal utility function. The vN-M expected utility theory interprets probability as objective, an alternative interpretation due to Savage treats probability as subjective. We will stick to the vN-M theory, the interested reader should consult Kreps (1988).

To simplify matters we will now assume that the space of consumption bundles is finite and call the elements of P are lotteries or gambles. A gamble can be written as sequence of probabilities $(\pi_1, \pi_2, \ldots, \pi_n)$ or $\{\pi_j\}_{j=1}^{n}$. In the finite case a preference relation on P has an expected utility representation if for gambles $p, q \in P$, $p \succeq q$ iff

$$\sum_x u(x)p(x) \geq \sum_x u(x)q(x)$$

The notion of an expected utility representation is important and represents a key to understanding the extent to which we may be able to quantify qualitative relationships. For the moment consider a preference relation qualitative: it captures without reference to any quantities the result of a certain decision making process. Expected utility is the mathematical average utility one will receive engaging in a given lottery. That a preference relation can be exactly captured by expected utility means among other things that a decision maker is governed ultimately by only those average utilities. This says something very definite about the nature of the preference relation, and possibly restricts its nature somewhat.

The rest of this section will be devoted to showing that a binary relation has an expected utility representation if the following three axioms are satisfied.

Axiom 1.3 $\succeq$ *is a preference relation on* P.

Axiom 1.4 (Continuity) $\succeq$ *is continuous. That is, if for all* $p, q, r \in P$ *the sets*

$$\{a \in [0,1] : ap + (1-a)q \succeq r\} \quad \textit{and}$$
$$\{a \in [0,1] : r \succeq ap + (1-a)q\}$$

are closed.

Axiom 1.5 (Independence) *For all* $p, q, r \in P$ *and* $a \in (0,1)$, *if* $p \succeq q$ *then* $ap + (1-a)r \succeq aq + (1-a)r$.

Continuity means that arbitrarily small changes in the probabilities don't change preferences between gambles. For example if you prefer going for a perfectly safe car ride to staying home, then you prefer going on a car ride with an arbitrarily small probability of an accident to staying home. Continuity rules out the lexicographic preference relation example above. This version of continuity guarantees the existence of an ordinal utility representation.

Independence says that if we mix two gambles with a third the resulting mixture does not depend on the third gamble. For example, suppose that your prefer betting \$10 on a hand of blackjack to betting \$10 on a spin of the roulette wheel. Now suppose that we flip a fair coin. Independence implies that you prefer a gamble of heads bet on blackjack, tails on craps to heads bet on roulette, tails on craps. In other words how you feel about craps should have no impact on your preference between the two compound gambles.

Theorem 1.6 (Expected Utility) *A preference relation on* P *(with finite alternative space) that satisfies the continuity and independence axioms admits an expected utility representation.*

Remark 1.3 For the theory to make sense a utility function must be bounded when probability distributions have infinite support. Suppose that u is unbounded from above. Then for every integer n there is some amount of money x_n such that $u(x_n) > 2^n$. Recall the St. Petersburg paradox game consists of tossing a fair coin repeatedly until it comes up heads. If the first head occurs on the n^{th} toss then the payout for the game is then $2n$. Since the probability of this outcome is 2^{-n} the expected utility of p is

$$\sum_{n=1}^{\infty} u(x_n)p(x_n) \geq \sum_{n=1}^{\infty} 2^n 2^{-n} = \infty$$

This means that you should be willing to pay an infinity amount of money (or at least your entire net worth) to take this gamble. How much would you pay?

1.4.1 Proof of expected utility theorem

We proceed with a series of lemmas.

Lemma 1.7 *If $\succeq$ satisfies the independence axiom , then for all $a \in (0,1)$ and $p, q, r \in P$ we have*

$$\begin{aligned} p &\succ q \Leftrightarrow ap + (1-a)r \succ aq + (1-a)r \\ p &\sim q \Leftrightarrow ap + (1-a)r \sim aq + (1-a)r \quad \text{and} \\ p &\succ q, r \succ s \Rightarrow ap + (1-a)q \succ ar + (1-a)s. \end{aligned}$$

Proof: Suppose $p \succ q$, then $p \succeq q$ and $q \not\succeq p$. by independence

$$\begin{aligned} ap + (1-a)r &\succeq aq + (1-a)r \quad \text{and} \\ aq + (1-a)r &\not\succeq ap + (1-a)r. \end{aligned}$$

Hence by the definition of $\succ$, $ap + (1-a)r \succ aq + (1-a)r$ □

Exercise 1.5 Complete the proof.

Proposition 1.8 *A utility function U has expected utility form if and only if it is a linear map from P to $\mathbb{R}$.*

Proof: Since X is finite and has vN-M utility u for convenience we can label $u(x_i) = u_i$ and then the expected utility is $\sum u_i \pi_i$. Suppose U is linear, i.e. for any $p_j \in P$ and $\sum_{j=1}^{n} a_j = 1$,

$$U\left(\sum_{j=1}^{n} a_j p_j\right) = \sum_{j=1}^{n} a_j U(p_j).$$

Let e_j be the degenerate gamble $\pi_i = 1$ if $i = j$, $\pi_i = 0$, $i \neq j$. We can write any $p = (\pi_1, \pi_2 \ldots, \pi_k)$ as a convex combination of the degenerate gambles $(e_1, e_2, \ldots, e_k)$ or $p = \sum_{i=1}^{k} \pi_i e_i$. We have

$$U(p) = U\left(\sum_i \pi_i e_i\right) = \sum_i \pi_i U(e_i) = \sum_i \pi_i u_i$$

Conversely, suppose that U has expected utility form, and consider any compound gamble $\{p_j\}_{j=1}^{n}$ with probabilities $\{a_j\}_{j=1}^{n}$ then

$$U\left(\sum_j a_j p_j\right) = \sum_i u_i \left(\sum_j a_j \pi_i^j\right) = \sum_j a_j \left(\sum_i u_i \pi_i^j\right) = \sum_j a_j U(p_j)$$

□

Exercise 1.6 Prove that an expected utility function u is unique only up to an affine transformation. In other words $u(x)$ and $v(x)$ represent the same preference relation iff $v(x) = au(x) + b$. (Hint: use the previous Proposition for the proof of necessity).

Exercise 1.7 Prove that there is best gamble p^* and a worst gamble p_* in P. (Hint: use the independence axiom).

Lemma 1.9 *If $p \succ q$ and $a \in (0, 1)$, then $p \succ ap + (1 - a)q \succ q$.*

Proof: Using Lemma 1.7

$$p = ap + (1 - a)p \succ ap + (1 - a)q \succ aq + (1 - a)q = q$$

□

Lemma 1.10 *If $a, b \in [0, 1]$ and $p^* \succ p_*$ then*

$$bp^* + (1 - b)p_* \succ ap^* + (1 - a)p_* \Leftrightarrow b > a$$

Proof: Set $c = (b - a)/(1 - a) \in (0, 1]$ then $bp^* + (1 - b)p_* = cp^* + (1 - c)(ap^* + (1 - a)p_*)$. Since $1 > b > a$, we know from Lemma 1.9 that $p^* \succ ap^* + (1 - a)p_*$ and then that $cp^* + (1 - c)(ap^* + (1 - a)p_*) \succ ap^* + (1 - a)p_*$. Hence $bp^* + (1 - b)p_* \succ ap^* + (1 - a)p_*$

Conversely suppose that $b \leq a$, if $b = a$ then $bp^* + (1 - b)p_* \sim ap^* + (1 - a)p_*$ so suppose $b < a$. Reversing the roles of b and a in the first part of the proof shows $ap^* + (1 - a)p_* \succ bp^* + (1 - b)p_*$. □

Lemma 1.11 *For all $p \in P$ there is a unique a_p such that $p \sim a_p p^* + (1 - a_p)p_*$.*

Proof: Consider the sets

$$\begin{array}{l} \{a \in [0,1] : ap^* + (1-a)p_* \succeq p\} \quad \text{and} \\ \{a \in [0,1] : p \succeq ap^* + (1-a)p_*\} \end{array}$$

Since $\succeq$ is continuous, both sets are closed and any a belongs to at least one set. Since both sets are nonempty and the unit interval is connected there must be some a that is in both sets. Call this a by a_p and observe that $p \sim a_p p^* + (1 - a_p)p_*$. Uniqueness follows from Lemma 1.10. $\square$
Proof of expected utility theorem: If $p^* \sim p_*$ then there is nothing to prove, so assume that $p^* \succ p_*$. Set $U(p) = a_p$, by Lemma 1.11 $p \succeq q$ if and only if $a_p p^* + (1 - a_P)p_* \succeq a_q p^* + (1 - a_q)p_*$, hence by Lemma 1.10 $p \succeq q$ if and only if $a_p \geq a_q$ and U represents a preference relation. It remains to show that U is linear. We have to show that for every $p, q \in P$ and $a \in [0,1]$

$$U(ap + (1-a)q) = aU(p) + (1-a)U(q)$$

By definition

$$\begin{array}{rcl} p & \sim & U(p)p^* + (1 - U(p))p_* \quad \text{and} \\ q & \sim & U(q)p^* + (1 - U(q))p_* \end{array}$$

Applying the independence axiom we have

$$\begin{array}{rcl} ap + (1-a)q & \sim & a\left(U(p)p^* + (1 - U(p))p_*\right) + (1-a)q \\ & \sim & a\left(U(p)p^* + (1 - U(p))p_*\right) + (1-a)\left(U(q)p^* + (1 - U(q))p_*\right) \\ & \sim & \left(aU(p) + (1-a)U(q)\right)p^* + \left(1 - aU(p) - (1-a)U(q)\right)p_* \\ & \sim & bp^* + (1-b)p_* \end{array}$$

where $b = aU(p) + (1-a)U(q)$. Therefore, by construction $U(ap + (1-a)q) = b = aU(p) + (1-a)U(q)$ $\square$

Remark 1.4 Under some fairly mild restrictions the theorem can be extend to X infinite and P the set of all borel probability measures. See Fishburn (1970).

1.4.2 Plausibility of the Axioms

From a positive perspective there is much empirical evidence that says the axioms of expected utility theory fail. See for example Kahneman and Tversky (1979). From a normative point of view the decision yours.

Definition 1.2 *For any* $x \in X$, *let* δ_x *denote the gamble that pays* x *with probability 1.*

Example 1.3 (The Allais Paradox) This is perhaps the most famous challenge to the axioms of expected utility, due to Allais (1953). We present the results of a particular experiment conducted by Kahneman and Tversky (1979). We consider gambles on three possible monetary outcomes, \$2.5 million, \$2.4 million, \$0. We denote by (π_1, π_2, π_3) the gamble with probability π_1 of winning \$2.5mm, π_2 of winning \$2.4mm and π_3 of winning \$0. Individuals are first ask to choose between the following two gambles, $p_1 = (.33, .66, .01)$ and $p_2 = (0, 1, 0)$. Take a moment and decide which you prefer. Then they are asked to choose between, $q_1 = (.33, 0, .67)$ and $q_2 = (0, .34, .66)$. Again, make the choice for yourself. Kahneman and Tversky observed that 82% of their survey group choose p_2 and 83% choose q_1, this implies that at least 65% choose p_2 and q_1. This is inconsistent with the preference relation having an expected utility representation. Suppose the individual has utility function u. $p_2 \succ p_1$ means

$$\begin{aligned} u(2.4) &> .33u(2.5) + .66u(2.4) + .01u(0) \\ .34u(2.4) &> .33u(2.5) + .01u(0) \end{aligned}$$

and $q_1 \succ q_2$ means

$$\begin{aligned} .33u(2.5) + .67u(0) &> .34u(2.4) + .66u(0) \\ .34u(2.4) &< .33u(2.5) + .01u(0) \end{aligned}$$

Clearly $.34u(2.4)$ cannot be both greater than and less than $.33u(2.5) + .01u(0)$. This can be viewed as violation of the independence axiom. A preference for p_2 over p_1 implies a preference for a sure \$2.4mm over a 33/34 chance \$2.5mm and a 1/34 chance of 0. I think most people would agree with this. The independence axiom implies that

$$.34\delta_{2.4} + .66\delta_0 \succ .34(\frac{33}{34}\delta_{2.5} + \frac{1}{34}\delta_0) + .66\delta_0$$

which is the opposite of $q_1 \succ q_2$. Kahneman and Tversky call this the "certainty effect", where individuals put "too much" weight on what is certain.

Example 1.4 (Machina's paradox) In this setting we have 3 outcomes: "a trip to Venice", "watching an excellent movie about Venice", and "staying home". Suppose that you prefer the first to the second and the second to the third. Now you are asked to choose between two gambles: $p_1 = (.999, .001, 0)$ and $p_2 = (.999, 0, .001)$. If you believe the independence axiom then you must choose p_1 over p_2. But you can easily see why you might choose the opposite. If you don't win the trip to Venice you might not want to be reminded of your misery by watching a movie about Venice.

Example 1.5 (Induced preferences) You are invited to a dinner party and you know that they will serve either lobster or steak with equal probability. You plan on bringing a bottle of wine to your host and would like to buy white if lobster is to be served and red if it is to be steak. Suppose that you are indifferent between lobster and steak and that red and white wine are both the same price. You must buy the wine before the party, i.e. the uncertainty will not be resolved until after the action is taken.

Since you like lobster and steak equally, you are apparently indifferent to a lottery that gives lobster with certainty and one that gives steak with certainty. According to the independence axiom you must also be indifferent to a lottery that gives lobster or steak each with probability 1/2. Clearly you are not indifferent since knowing the outcome with certainty would allow you to buy the right wine.

We must be careful here, this example does *not* violate the independence axiom. To use the theory properly we must set up the decision framework to the utility derived from an outcome does not depend on any action taken by the individual before the uncertainty is resolved. In other words preferences should not be induced from ex ante actions. To use the theory properly we must include the ex ante action as part of the outcomes. In this example the outcomes might be,

1. Bring white wine and lobster is served.
2. Bring white wine and steak is served.
3. Bring red wine and lobster is served.
4. Bring red wine and steak is served.

Now we can consider a probability distribution on these 4 outcomes and there is no reason that independence axiom must be violated.

Partly because of these examples there has been much research on a theory of choice under uncertainty that relaxes the independence axiom. Nevertheless, most of the theory of choice that you need for finance (and economics) is based on the vN-M expected utility model.

1.5 Wealth and Risk Aversion

In general we consider a world with many consumption goods and many time periods. We will now consider the special case of one good and one time period. In this world consumption and wealth are equivalent. In the multiperiod setting taking wealth as a proxy for consumption is not adequate even if there is a single consumption good. This is because we may not decide to consume all of our wealth at the end of each period. Nevertheless, we now take P to be the set of probability distributions on $\mathbb{R}$ or $\mathbb{R}_+$. Later we will see that we can derive many useful conclusions with this simplified model. As alluded to in Remark 1.4, we can extend the expected utility theorem to this case.

There are many different notations for expected utility. If we have a probability distribution p on the line with cumulative distribution function F and perhaps a probability density function f we can write

$$E^p[u] \equiv \int u(x)dp(x) \equiv \int u(x)p(dx) \equiv \int u(x)dF \equiv \int u(x)f(x)dx$$

so don't let this confuse you. We will write $\mu(p)$ for the expected value of the gamble p, $\int xdp(x)$.

We will assume that u is increasing(i.e. more is preferred to less), continuous, and bounded since this makes intuitive sense when talking about utility of money.

Definition 1.3 *Preferences are said to be risk averse or to exhibit risk aversion if $\delta_{\mu(p)} \succeq p$ for all $p \in P$, strictly risk averse if $\delta_{\mu(p)} \succ p$ for all $p \in P$ such that the variance of p is not 0, and risk neutral if $\delta_{\mu(p)} \sim p$ for all $p \in P$. Analogous definitions for risk seeking and strictly risk seeking are self evident.*

If the preference relation admits an expected utility representation $u(x)$ then it follows directly from the definition that the individual is risk averse if and only if

$$\int u(x)dp(x) \le u\left(\int xdp(x)\right) \quad \text{for all } p$$

This inequality is called Jensen's inequality and is the defining property of a concave function. Hence in the case of expected utility theory risk aversion is equivalent to concavity of the utility function. Similarly preferences are risk seeking if and only if u is convex and risk neutral iff u is affine.

Definition 1.4 *The certainty equivalent for some utility function u and probability distribution p, $c(p,u)$ is the amount of money it takes to make the individual indifferent between the gambles p and $\delta_{c(p,u)}$*

$$u(\delta_{c(p,u)}) = \int u(x)dp(x)$$

Definition 1.5 *We define the risk premium of u and p as*

$$\lambda(p,u) = \int xdp(x) - c(p,u)$$

Proposition 1.12 *Equivalent are:*

1. The individual is risk averse.
2. u is concave.
3. $c(p,u) \le \int xdp(x)$ for all p.

Proof: It remains only to prove the last part. Since u is increasing

$$c(p,u) \le \int xdp(x) \Leftrightarrow u\left(\delta_{c(p,u)}\right) \le u\left(\int xdp(x)\right) \Leftrightarrow \int u(x)dp(x) \le u\left(\int xdp(x)\right)$$

□

We will assume for convenience that u is twice continuously differentiable from here on in. It is often convenient to have a measure of risk aversion. Its seems reasonable that risk aversion is related to the curvature of u. One measure of curvature is $u''(x)$, but this measure is not scale invariant. We make it invariant by dividing by u' and we change the sign to give a positive number for more risk aversion.

Definition 1.6 *The Arrow-Pratt measure of absolute risk aversion at x for an individual with utility function u is*

$$R_a(x,u) = \frac{-u''(x)}{u'(x)}$$

Observe that by two integrations we can recover u from $R_a(x,u)$ up to two constants of integration, but since utility is only unique up to affine transformations R_a completely characterizes u.

Exercise 1.8 Suppose $R_a(x,u) = $ constant. What two parameter family of utility functions does R_a characterize.

We want to compare two individuals with utility functions u_1 and u_2 and ask when u_1 is more risk averse than u_2.

Definition 1.7 *u_1 is said to be at least as risk averse as u_2 if for all $p \in P$, $\lambda(p,u_1) \geq \lambda(p,u_2)$.*

Proposition 1.13 *Equivalent are:*

1. u_1 is at least as risk averse as u_2.
2. $R_a(x,u_1) \geq R_a(x,u_2)$ for all x.
3. The is an increasing concave function g such that $u_1 = g(u_2(x))$ for all x. I.e. u_1 is more concave than $u_{2.}$
4. $c(p,u_1) \leq c(p,u_2)$ for all p.
5. $\int u_1(x)dp(x) \geq u_1(y)$ implies $\int u_2(x)dp(x) \geq u_2(y)$

Proof: See Kreps (1988). □

Definition 1.8 *We say that u exhibits decreasing absolute risk aversion if for all $p \in P$, $x, w_1, w_2 \in \mathbb{R}$, such that $w_2 > w_1$*

$$E^{w_1+p}[u] > u(w_1+x) \text{ implies } E^{w_2+p}[u] > u(w_2+x)$$

where $w+p$ is the gamble p with each outcome shifted by the amount w.

In other words the wealthier the individual the more risk she takes in absolute dollar terms.

1. u exhibits decreasing absolute risk aversion.

2. $R_a(x, u)$ is a decreasing function of x.

3. For $x_1 < x_2$, $u_1(z) = u_1(x_1 + z)$ is a concave transformation of $u_2(z) = u_2(x_2 + z)$.

4. $\lambda(p + w, u)$ is an increasing function of w.

Proof: Kreps (1988). □

Absolute risk aversion is suited to gambles whose outcomes are absolute gains or losses from current wealth. For gambles whose gains and losses are measured as a proportion of wealth we define the concept of relative risk aversion. Let α represent proportional increments of wealth. Consider the utility function on wealth w, $v(\alpha) = u(\alpha w)$. Initial wealth is $\alpha = 1$. Near $\alpha = 1$ risk aversion is measured as $\frac{v''(1)}{v'(1)} = w\frac{u''(w)}{u'(w)}$.

Definition 1.9 *The Arrow-Pratt measure of relative risk aversion at x for an individual with utility function u is*

$$R_r(x, u) = \frac{-xu''(x)}{u'(x)}$$

Exercise 1.9 For each of the following utility functions calculate $R_a(x, \cdot)$, $R_r(x, \cdot)$. State whether each utility function is increasingly or decreasingly risk averse.

$$\begin{aligned} u_1(x) &= x - \frac{b}{2}x^2, \qquad b > 0 \qquad \text{(Quadratic Utility)} \\ u_2(x) &= \frac{1}{1-\gamma}x^{1-\gamma} \qquad x > 0, \gamma > 1 \qquad \text{(Power Utility)} \\ u_3(x) &= -e^{-ax} \qquad a \geq 0 \qquad \text{(Negative Exponential Utility)} \end{aligned}$$

1.6 References

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